

THE CUBICAL COMPLEX OF A PERMUTATION GROUP REPRESENTATION – OR HOWEVER YOU WANT TO CALL IT

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ABSTRACT. This paper is about a small combinatorial trick, which is well known, but has no name. Let G be a permutation group acting on a vector space M . There is a natural way to assign a cosimplicial space to these data. We call the resulting cochain complex the cubical complex. Its cohomology is easy to compute. We give some examples of its occurrence in nature.

1. A GENERAL COMBINATORIAL FACT

Let k be a field of characteristic zero, $n \in \mathbb{N}$ and $A = k[x_1, \dots, x_n]$ the polynomial co-algebra. Let $C(A)$ be its bar complex, computing $Tor_A(k, k)$. Since A is Koszul it is well known that the homology $H(C(A)) = Tor_A(k, k) \cong k[\epsilon_1, \dots, \epsilon_n]$ is the Koszul dual algebra, where ϵ_j are odd (degree one) variables.

On the complex $C(A)$ there is a \mathbb{N}_0^n -grading by the numbers of x_1, \dots, x_n occurring. Let $C(A)^{(1, \dots, 1)}$ be the degree $(1, \dots, 1)$ subcomplex. For instance, if $n = 3$ then $x_1 \otimes x_2 x_3$ is inside this subcomplex, while $x_1 \otimes x_2 x_1$ is not. It follows from the above that

Proposition 1.

$$H(C(A)^{(1, \dots, 1)}) = (k[\epsilon_1, \dots, \epsilon_n])^{(1, \dots, 1)} = k \epsilon_1 \dots \epsilon_n \cong k[-n].$$

Remark 1. As noted by V. Drinfeld [4], the complex $C(A)^{(1, \dots, 1)}$ is the cosimplicial complex of an n dimensional hypercube, relative to its boundary.

Note further that $C(A)^{(1, \dots, 1)} \cong \bigoplus_m (k^m)^{\otimes n}$, m being the degree. Note also that the differential is invariant under the S_n action by permuting the factors. Since taking (co-)invariants under finite group actions commutes with taking homology, we obtain:

Corollary 1. *Let $G \subset S_n$ be a subgroup and M some right G -module. Then*

$$H(M \otimes_G C(A)^{(1, \dots, 1)}) = M \otimes_G \text{sgn}_n$$

where sgn_n is the one dimensional sign representation of G , concentrated in degree n .

Definition 1. We call the complex $M \otimes_G C(A)^{(1, \dots, 1)}$ the *cubical complex* $Cub(G, M)$ of the pair (G, M) .

Remark 2. There is a similar complex based on the Harrison complex $Harr(A)$, namely

$$Harr(G, M) = M \otimes_G Harr(A)^{(1, \dots, 1)},$$

which we call the Harrison complex of the pair (G, M) . It is well known that $H(Harr(A)) \cong k^n[-1]$. Hence the following is also well known:

$$H(Harr(G, M)) \cong \begin{cases} 0 & \text{for } n > 1 \\ M[-1] & \text{for } n = 1. \end{cases}$$

Remark 3. It is actually sufficient to consider the case $G = S_n$. If $G \subsetneq S_n$, one can equivalently take $G' = S_n$, $M' = M \otimes_G k[S_n] = \text{Ind}_G^{S_n} M$.

2. EXAMPLES

Let us take $G = S_n$ and $M = Lie(n)$ the n -ary operations in the Lie operad. By definition,

$$\begin{aligned} \bigoplus_n Cub(S_n, Lie(n)) &= \bigoplus_n Lie(n) \otimes_{S_n} C(A)^{(1, \dots, 1)} = \\ &= \bigoplus_{m,n} Lie(n) \otimes_{S_n} (k^m)^{\otimes n} = \bigoplus_m \text{FreeLie}(k^m) =: \text{lie} \end{aligned}$$

is a complex built from the free Lie algebras, which one can check is the same as the one considered by A. Alekseev and C. Torossian [1]. One hence obtains:

Corollary 2.

$$H(\mathfrak{lie}) = \oplus_n \text{Lie}(n) \otimes_{S_n} \text{sgn}_n = k[-1] \oplus k[-2].$$

Proof. The first equality is the previous Corollary, second follows from the fact that there are no fully antisymmetric elements in $\text{Lie}(n)$ for $n \geq 3$ by the Jacobi identity. \square

This cohomology was computed by M. Vergne [5] (the second part of Thm 2.3 in [5]).

A simpler and very similar example is $G = S_n$ and $M = k[S_n] = \text{Ass}(n)$, the space of n -ary operations in the associative operad. In this case $\oplus_n \text{Cub}(S_n, k[S_n])$ is a complex built from free associative algebras and its cohomology is $\oplus_n k[S_n] \otimes_{S_n} \text{sgn}_n = \oplus_n k[-n]$. This is the first part of Theorem 2.3 in [5].

Next, take again $G = S_n$ and $M = k[S_n]_{C_n}$ where $C_n \subset S_n$ is the cyclic group. Equivalently, $M = \text{Ass}(n-1)$, where we view Ass as a cyclic operad. In this case we get the complex \mathfrak{tr} defined by Alekseev and Torossian [1]

$$\begin{aligned} \mathfrak{tr} &= \oplus_n k[S_n]_{C_n} \otimes_{S_n} C(A)^{(1, \dots, 1)} \cong \oplus_{m,n} k[S_n]_{C_n} \otimes_{S_n} (k^m)^{\otimes n} = \oplus_{m,n} ((k^m)^{\otimes n})_{C_n} \\ &= \oplus_m \text{FreeAss}(k^m) / [\text{FreeAss}(k^m), \text{FreeAss}(k^m)]. \end{aligned}$$

Applying Corollary 1 we get

Corollary 3.

$$H(\mathfrak{tr}) = \oplus_n k[S_n]_{C_n} \otimes_{S_n} \text{sgn}_n = \oplus_n k \otimes_{C_n} \text{sgn}_n \cong \oplus_{n \text{ odd}} k[-n]$$

We can also take $G = S_n$ and $M = \text{Lie}(n-1)$. This is possible since Lie (-as a suboperad of Ass -) is a cyclic operad. Consider

$$\oplus_n \text{Lie}(n) \otimes_{S_{n+1}} C(A)^{(1, \dots, 1)} = \oplus_{m,n} \text{Lie}(n) \otimes_{S_{n+1}} (k^m)^{\otimes n} = \oplus_m \mathfrak{sdet}_m =: \mathfrak{sdet}.$$

Note that this space coincides with the total space of the operad of Lie algebras \mathfrak{sdet} defined in [1], which in turn is the same as the \mathcal{F} defined in [4]. By a similar argument as before, we hence obtain:

Corollary 4.

$$H(\mathfrak{sdet}) = \oplus_n \text{Lie}(n) \otimes_{S_{n+1}} \text{sgn}_{n+1} \cong k[-3].$$

Remark 4. All examples above have analogs for their respective Harrison complexes. Concretely, the cohomology in this case is always given by the $n = 1$ -part of the cohomology computed in the above propositions. We leave the details to the reader.

3. GENERALIZATIONS AND FURTHER EXAMPLES IN THE LITERATURE

There is a generalization to the above trick. Let \mathcal{P} be any cooperad and A the free \mathcal{P} -coalgebra on k^n . Then A is endowed with a \mathbb{Z}^n -grading, along with an S_n action. Any “sensible” homology theory of \mathcal{P} -coalgebras will produce a complex $C(A)$, inheriting this grading and S_n action. For any right S_n module M one can hence form the complex

$$M \otimes_{S_n} C(A)^{(1, \dots, 1)}$$

with cohomology $M \otimes_{S_n} H(C(A))^{(1, \dots, 1)}$. This situation occurs frequently in nature, for example, taking for \mathcal{P} the coassociative cooperad and $C(A)$ the cyclic complex, one can produce a proof of Theorem 8 in [2]. Taking $P = e_2^*\{2\}$ the co-Gerstenhaber cooperad (up to degree shift), and for $C(A)$ the Gerstenhaber cohomology, one can produce a shorter proof of Proposition 21 in [6].

We want to emphasize that the combinatorial trick described here is not novel, and has been used many times in the mathematical literature, in various forms and by various authors.¹ This paper was written merely to advertise a clean version of the trick and point out some further applications.

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¹In particular, D. Bar-Natan made us aware of his article [3], where the case of general M is discussed as well.